Changing the Particle Energy

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Why Accelerate Particles?

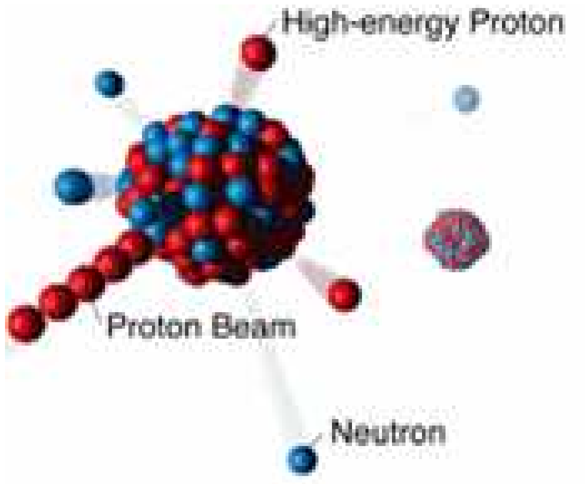
In both the relativistic and non-relativistic case "accelerating" a particle means to modify (increase) its energy.

In most of accelerator applications the particle energy is one of the fundamental design parameter and tuning knob:

- **In colliders tuning the c.m. energy on resonance allows to create new particles**
- **In light sources the energy defines the spectrum of the emitted radiation**
- **The energy defines the penetration depth of a particle inside materials (cancer therapy, …)**

In relativistic particles storage rings the energy losses need to be restored in order to keep the particles stored.

How to Accelerate Particles?



Neutral particles can be accelerated by: scattering, 'spallation'

Charged particles: Electric fields mainly.

Much more efficient and much more controllable

Large number of schemes and techniques used to generate the required electric fields.
Continuous R&D going on

Electromagnetic Fields

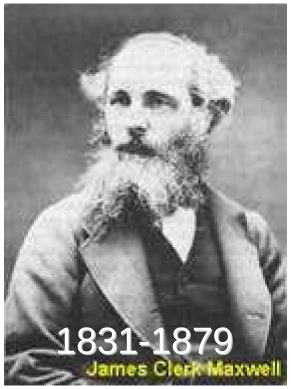
Maxwell Equations in vacuum (SI Units – differential form):

$$\nabla \cdot \overline{E} = \frac{\rho}{\varepsilon_0}$$ Coulomb's or Gauss' law for electricity

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$ Faraday's law

$$\nabla \cdot \overline{B} = 0$$ Gauss' law for magnetism

$$\nabla \times \overline{B} = \mu_0 \overline{J} + \mu_0 \varepsilon_0 \frac{d\overline{E}}{dt}$$ Ampere's law



Time variable magnetic fields are <u>always</u> associated with electric fields (and vice versa)

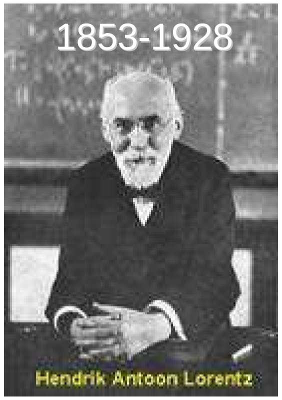
Lorentz Equation

$$\overline{F} = q\left(\overline{E} + \overline{v} \times \overline{B}\right)$$

$$W = \int \overline{F} \cdot d\overline{l} = q\int \overline{E} \cdot d\overline{l} + \cancel{q\int \left(\overline{v} \times \overline{B}\right) \cdot d\overline{l}}$$

B fields can change the trajectory of a particle
But <u>cannot</u> do *work* and thus change its energy

$$\boxed{\overline{F} = q\overline{E} \qquad W = q\int \overline{E} \cdot d\overline{l}}$$



Electric Field Representation

Plane wave representation:

$$\overline{E} = \overline{E}_o e^{i(\omega t - ks)} = \overline{E}_o \left[\cos(\omega t - ks) + i \sin(\omega t - ks)\right]$$

We will use in our calculations this representation.
Such a representation is quite general.
In fact, arbitrary electric fields can be represented as:

$$\overline{E} = \sum_{n=-\infty}^{\infty} \overline{E}_{no} e^{i(n\omega_0 t - ks)} \qquad \overline{E} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ f(\omega) e^{i(\omega t - ks)}$$

Periodic Case — Non-periodic Case

From DC to ….

In actual accelerators we often deal with a single frequency:

$$\overline{E} = \overline{E}_0 e^{i(\omega t - ks)}$$

$\omega/2\pi \approx 0$ **Electrostatic Accelerators**

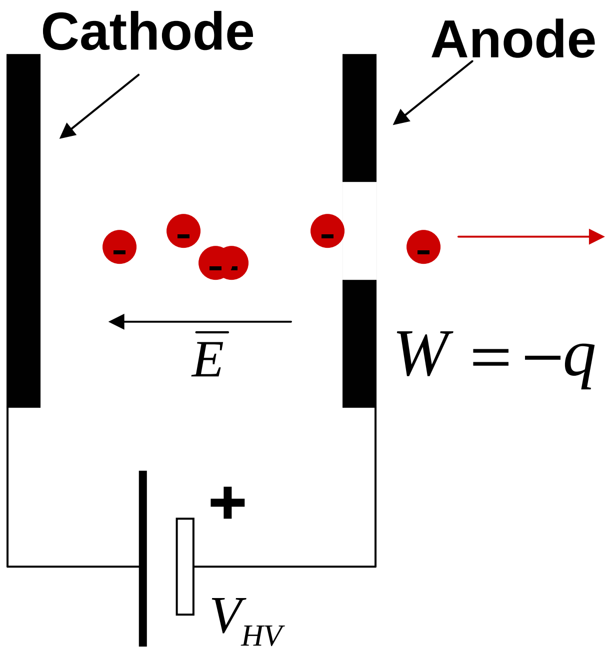
$\omega/2\pi \approx 10 - 10^3\ Hz$ **Induction, Betatrons**

Present dominant technology

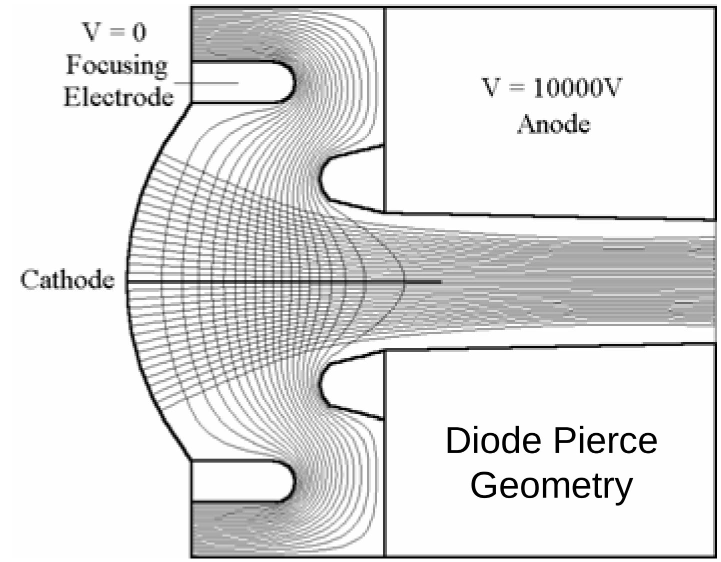
$\omega/2\pi \approx 10^6 - 10^{11}\ Hz$ **Radio Frequency (RF) accelerators**

$\omega/2\pi \approx 10^{12} - 10^{18}\ Hz$ **Laser ponderomotive accel.**

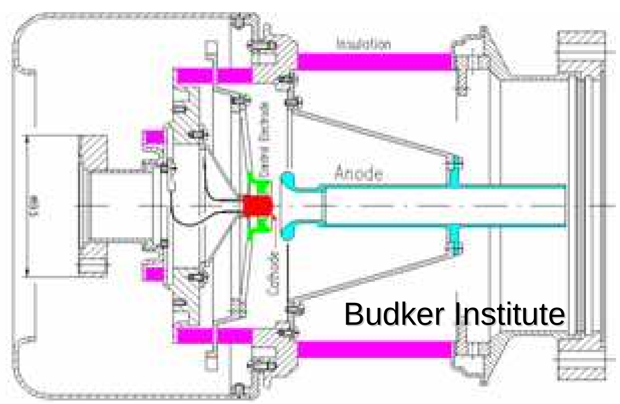
Electrostatic Accelerators: The Simplest Scheme

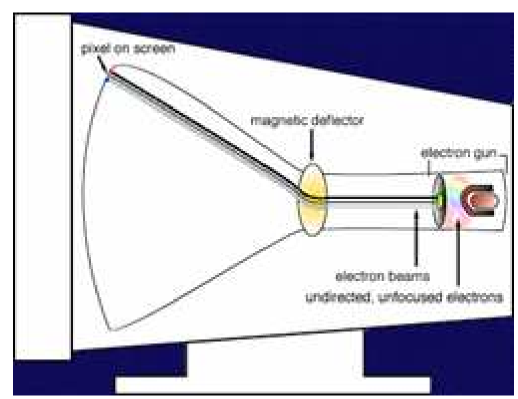


$$W = -q\, V_{HV}$$

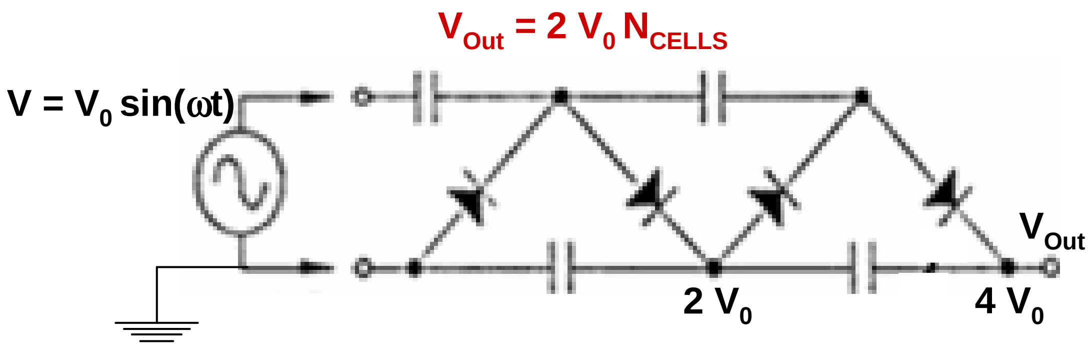


Still one of the most used schemes for electron sources





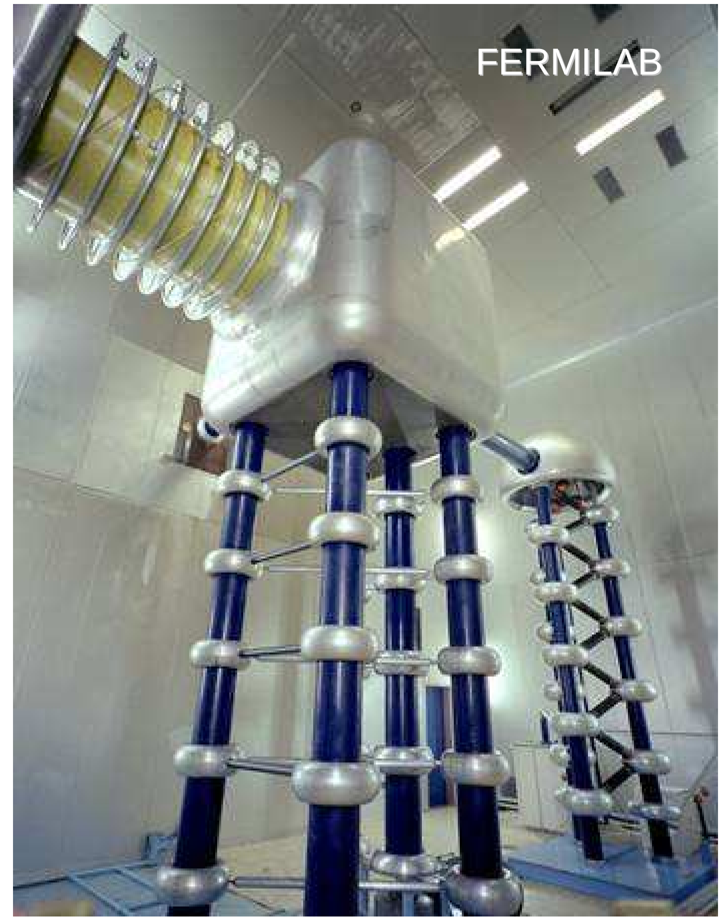
Electrostatic Accelerators: The Cockcroft-Walton Scheme



CASCADE GENERATOR (1914)

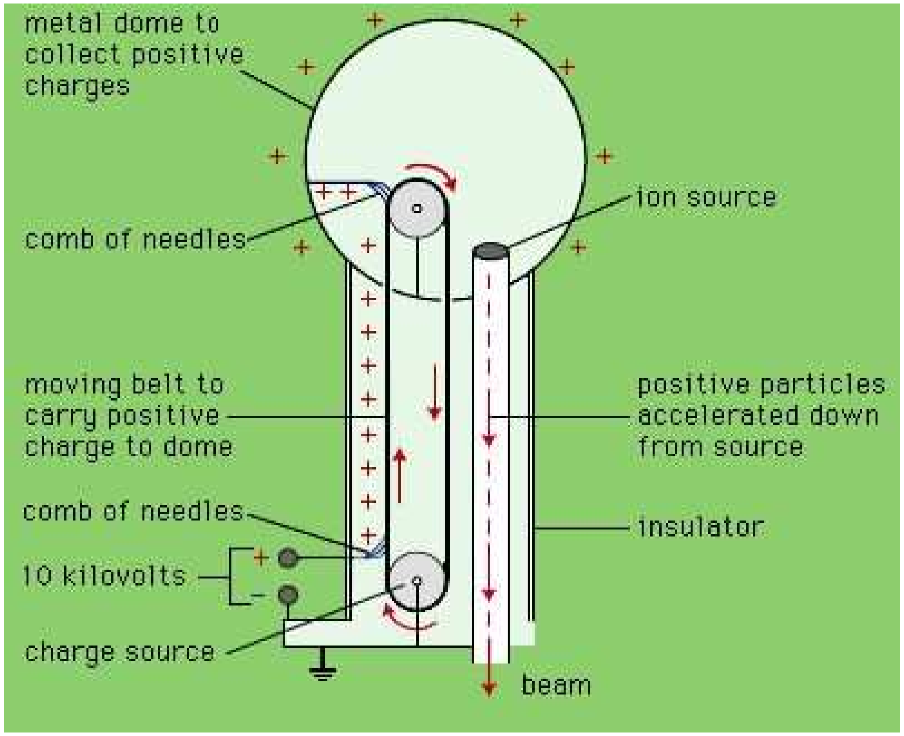
James Cockcroft and Ernest Walton in 1932 accelerated protons to 800 keV and produced fission of Lithium in Helium (Nobel Prize 1951)

$$p + Li \rightarrow 2He$$

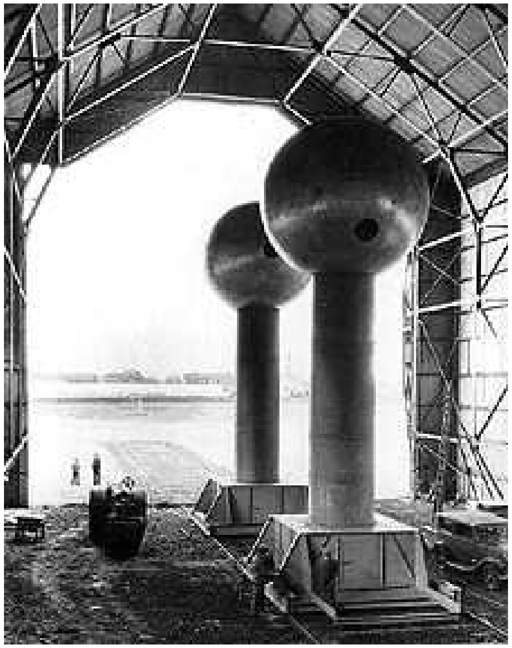


Still used as the first accelerator stage for protons and ions

Electrostatic Accelerators: The Van de Graaff



- The needle transmits the charge to the belt by glow discharge and/or field emission
- The electric field inside the sphere is zero permitting the passage of the charge from the belt to the sphere

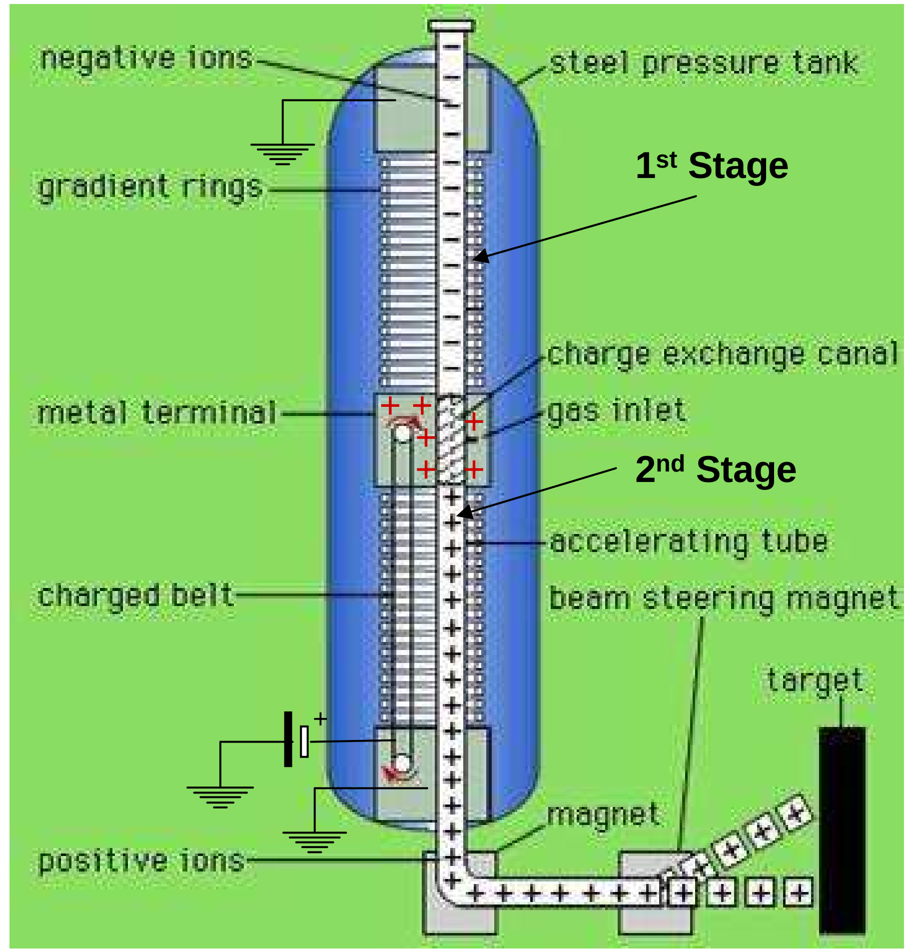


- The maximum voltage is limited by voltage breakdown. Inert gasses (Freon, SF6) help.

7MV in 1933
~ 20 MV nowadays

Van de Graaff Accelerator: Applications

Tandem Scheme



•Negative ions (H^- for example) are created and accelerated through the first stage

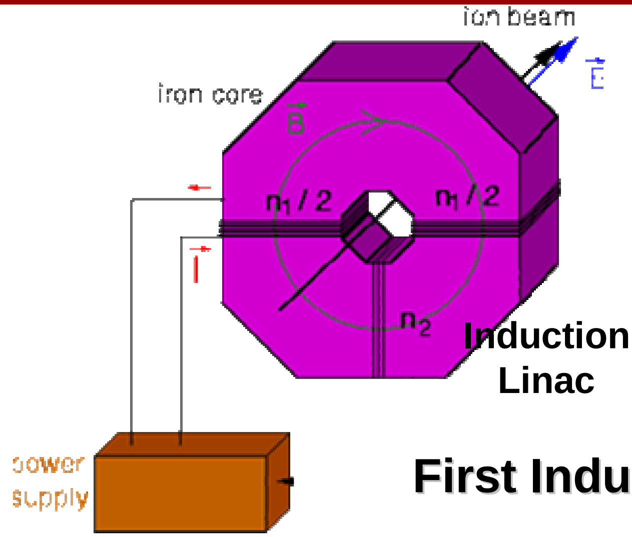
•At the end of the first stage the electrons are 'stripped' out from the ions (by a gas target for example)

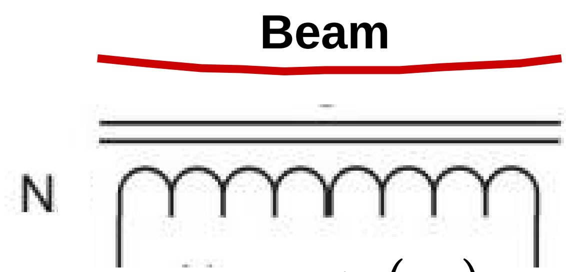
•In the second stage the positive ions (protons in our example) are accelerated. The net energy gain is <u>twice</u> the voltage of the Van de Graaff

Low Frequency Accelerators: Induction Linacs & Betatrons

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$

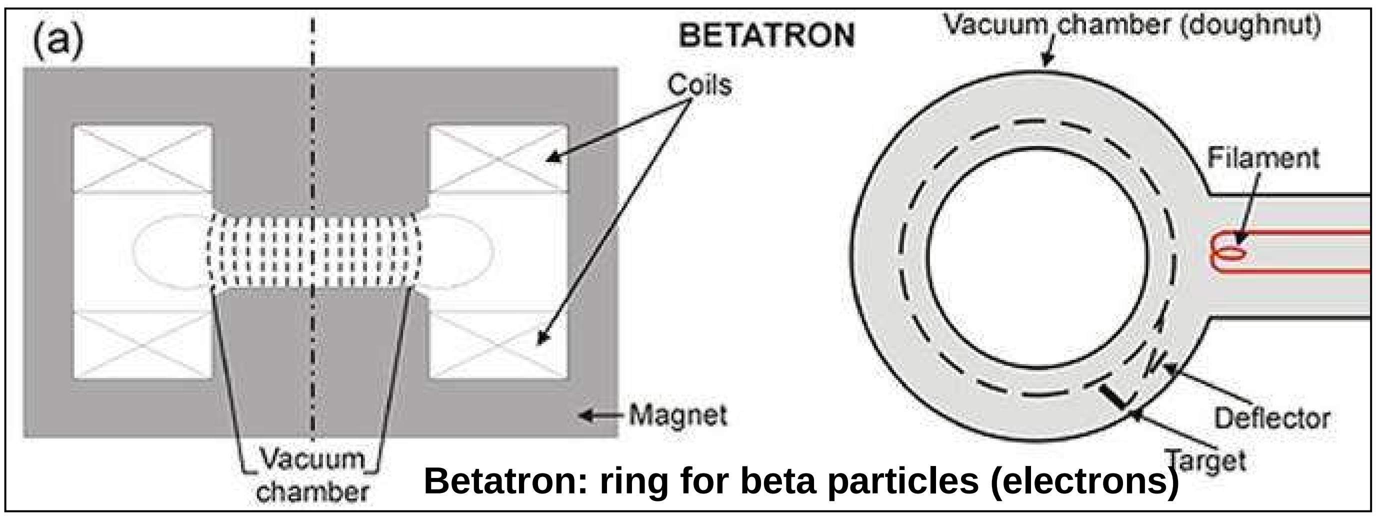
$$\omega / 2\pi \approx 50 \div 60 \ \text{Hz}$$





$$I = I_0 \sin(\omega t)$$

First Induction Accelerators in ~ 1935



Betatron: ring for beta particles (electrons)

Induction Accelerators allows for very high currents (~ 1kA) at relatively moderate energies (few MeV)

EM Fields in Free Space: The Wave Equation

From Maxwell equations it is possible to derive for free space electromagnetic waves:

$$\boxed{\nabla^2 E_i = \frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} \qquad i = x, y, z}$$

$Laplacian:$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For accelerating particles we need a non-zero field component in the z direction (for example) ➡

$$\nabla^2 E_z = -\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

And we will look for a solution in the shape:

$$E_z = E_{0z}(x,y)\left[A_F\left(e^{i(\omega t - kz)} \pm e^{-i(\omega t - kz)}\right) + A_B\left(e^{i(\omega t + kz)} \pm e^{-i(\omega t + kz)}\right)\right]$$

Forward wave

Backward wave

EM Fields in Free Space: Solution in Cylindrical Coordinates

The typical RF accelerating structures present axial symmetry. The natural coordinates for this case are the cylindrical coordinates.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad + \quad E_z = E_{0z}(r,\theta)e^{\pm i\omega t}e^{\pm ikt} \quad + \quad \nabla^2 E_z = -\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 E_{0z}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{0z}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_{0z}}{\partial \theta^2} + \left(\frac{\omega^2}{c^2} - k^2\right)E_{0z} = 0$$

Again, because of the axial symmetry it is convenient to assume that the azimuthal component of the field has periodicity *n*

$$E_{0z}(r,\theta) = \tilde{E}_{0z}(r)e^{\pm in\theta} \quad \Rightarrow \quad \frac{\partial^2 \tilde{E}_{0z}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_{0z}}{\partial r} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{n^2}{r^2}\right)\tilde{E}_{0z} = 0$$

Which has as general solution:

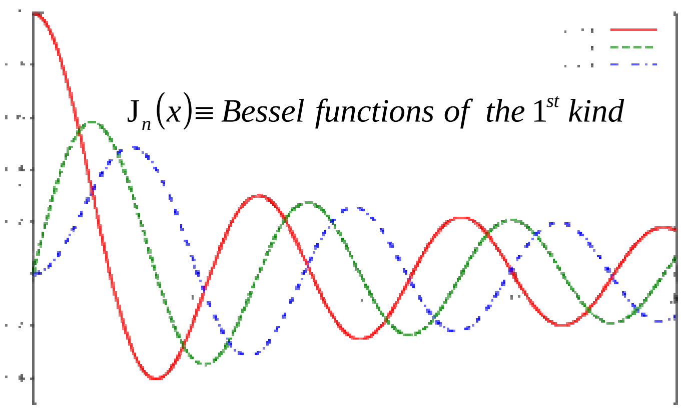
$$\boxed{\tilde{E}_{0z} = A\,\mathrm{J}_n(k_C r) + B\,\mathrm{Y}_n(k_C r)}$$

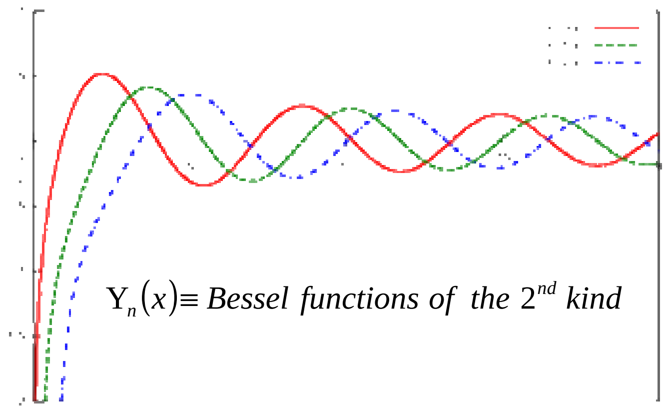
where

$$\boxed{k_C^2 = \frac{\omega^2}{c^2} - k^2 \quad \text{cutoff wavenumber}}$$

EM Fields in Free Space: Wave Equation General Solution

$$E_z = \left[A \mathrm{J}_n(k_C r) + \cancel{B \mathrm{Y}_n(k_C r)}\right] e^{\pm i n\theta} e^{\pm i\omega t} e^{\pm ikz}$$





$$E_z = \mathrm{J}_n(k_C r)\cos(n\theta)\left[A_F\left(e^{i(\omega t - kz)} \pm e^{-i(\omega t - kz)}\right) + A_B\left(e^{i(\omega t + kz)} \pm e^{-i(\omega t + kz)}\right)\right]$$

And analogously for the longitudinal component of the magnetic field:

$$B_z = \mathrm{J}_n(k_C r)\cos(n\theta)\left[C_F\left(e^{i(\omega t - kz)} \pm e^{-i(\omega t - kz)}\right) + C_B\left(e^{i(\omega t + kz)} \pm e^{-i(\omega t + kz)}\right)\right]$$

By using these expressions and the Maxwell curl equations it is possible to derive similar expressions for E_r, E_θ, B_r and B_θ

EM Fields in Free Space: Boundary Conditions & Classification

The general solution can now be applied to real geometries.

By imposing the values that the fields assume at the boundaries (boundary conditions), the values of the constants in the general solution can be evaluated and the solution for the specific geometry is found.

It is useful to classify the possible solutions in the following categories:

- **TEM modes**: where both the electric and magnetic fields are in the transverse plane
- **TE modes**: where the electric field is in the transverse plane
- **TM modes**: where the magnetic field is in the transverse plane ⟺ Good for Acceleration!

$TMn_\theta n_r n_z$

n_θ : periodicity in θ

n_r : periodicity in r

n_z : periodicity in z

EM Fields in Free Space: The Cutoff Frequency

From the definition of cutoff wavenumber:

$$k^2 = \frac{\omega^2}{c^2} - k_C^2$$

By defining:

$$\boxed{\omega_C = c k_C \quad \textbf{Cutoff (angular) frequency}}$$

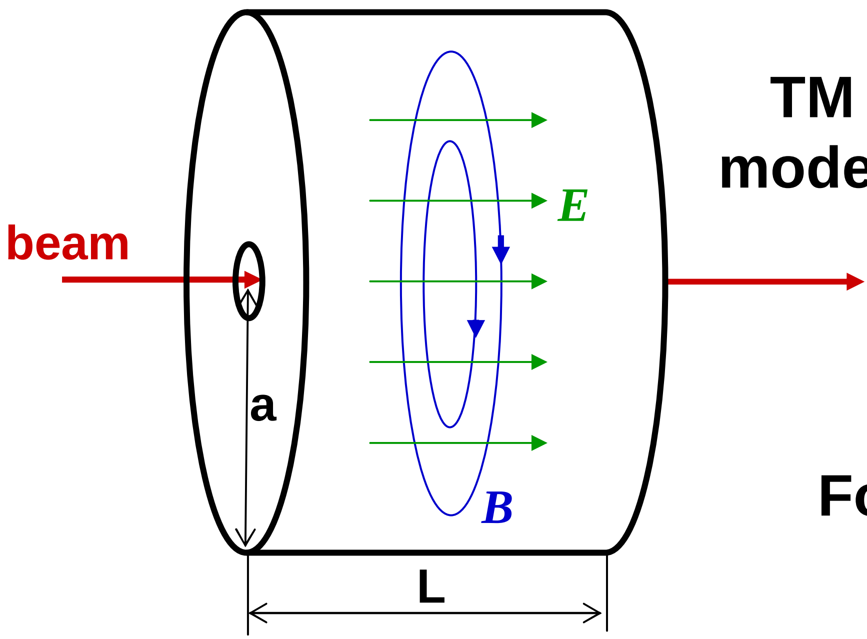
$$E_z = J_0(k_C r) A_F \, e^{i(\omega t - kz)}$$

$$\omega > \omega_C \Rightarrow k^2 > 0 \Rightarrow k \text{ is real} \Rightarrow \text{the wave propagates}$$

$$\omega < \omega_C \Rightarrow k^2 < 0 \Rightarrow k \text{ is immaginary} \Rightarrow \begin{array}{l}\text{the wave does not propagate} \\ \text{and decreases exponentially}\end{array}$$

$$\omega = \omega_C \Rightarrow k = 0 \Rightarrow \text{the wave does not propagates and does not depend on } z$$

EM Fields in Free Space: Pill Box Cavity Example



TM modes

Pill Box Boundary Conditions:

$$E_z(r=a)=E_\theta(r=a)=0$$

$$E_r(z=0)=E_\theta(z=0)=0$$

$$E_r(z=L)=E_\theta(z=L)=0$$

For example, for the TM_{010} mode:

$$\partial E_z^{TM010}/\partial z=\partial E_z^{TM010}/\partial\theta=0$$

$$E_z^{TM010}=E_0 J_0(k_C r)\cos(\omega_C t) \qquad k_C=2.405/a$$

$$B_\theta^{TM010}=-\frac{\omega}{c^2 k_C}E_0 J_1(k_C r)\sin(\omega_C t) \qquad f_C=\omega_C/2\pi=2.405c/2\pi a$$

$$\text{Example:}\quad f_0=500\,MHz \iff a=229.5\,mm$$

Phase Velocity & Group Velocity: Definitions

$$E_z = E_0 \cos(\omega t - kz) \qquad \varphi = \omega t - kz$$

$$\frac{d\varphi}{dt} = \omega - k\frac{dz}{dt} = 0$$

$$\boxed{v_P = \frac{\omega}{k} \qquad \textbf{Phase Velocity}}$$

$$v_P = \frac{\omega}{k} = \frac{\omega}{\sqrt{(\omega/c)^2 - k_C^2}} = \frac{\omega}{\sqrt{(\omega/c)^2 - (\omega_C/c)^2}} = \frac{c}{\sqrt{1-(\omega_C/\omega)^2}} > c$$

For propagating waves $v_P > c$ ➡ **No acceleration is possible!**

Group Velocity

$$v_G = \frac{d\omega}{dk}$$

$$\omega = c\sqrt{k_C^2 + k^2}$$

$$v_G = \frac{d\omega}{dk} = \frac{ck}{\sqrt{k_C^2 + k^2}} = \frac{c}{\sqrt{1+k_C^2/k^2}} < c$$

Phase Velocity & Group Velocity Physical Meaning

$$E_z = E_0 \cos(\omega t - kz)$$

$$v_P = \frac{\omega}{k} > c$$

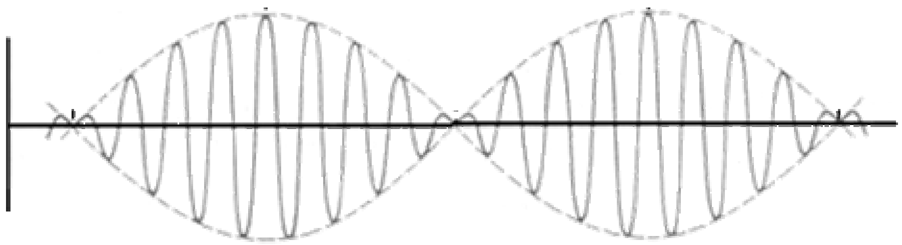
v_P is the speed at which the phase propagates or at which the wave propagates rigidly

$$v_G = \frac{d\omega}{dk} < c$$

v_G is the speed at which the energy propagates or a variation of the wave envelope propagates

For example: wave beating.

$$\cos[(\omega+\Delta\omega)t-(k+\Delta k)z]+\cos[(\omega-\Delta\omega)t-(k-\Delta k)z]=2\cos(\omega t-kz)\cos(\Delta\omega t-\Delta kz)$$

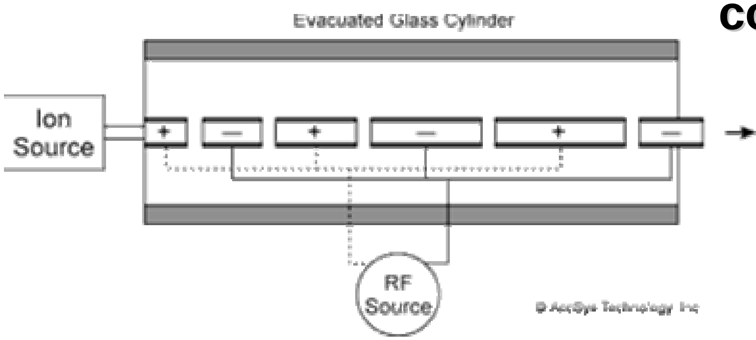


$$v_{P\omega} = \frac{\omega}{k} \qquad v_{P\Delta\omega} = \frac{\Delta\omega}{\Delta k} \approx \frac{d\omega}{dk}$$

RF Accelerators: Wideroe and Alvarez Schemes

In 1925-28 Ising and Wideroe conceived the first linear accelerator (linac). The revolutionary device was based on the *drift tubes scheme*.

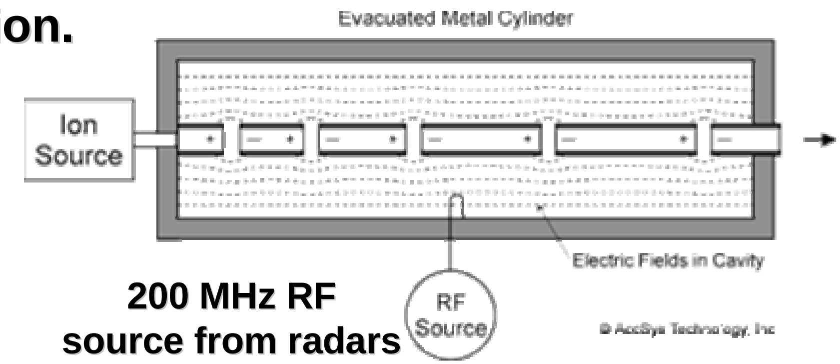
During the decelerating half period of the RF, the beam is shielded inside the conductive tubes.



Synchronicity condition: $L_i \cong \frac{1}{2} v_i T_{RF}$

At high frequency the Wideroe scheme becomes lossy due to electromagnetic radiation.

In 1946 Alvarez overcame to the inconvenient by including the Wideroe structure inside a large metallic tube forming an efficient cavity where the fields were confined.

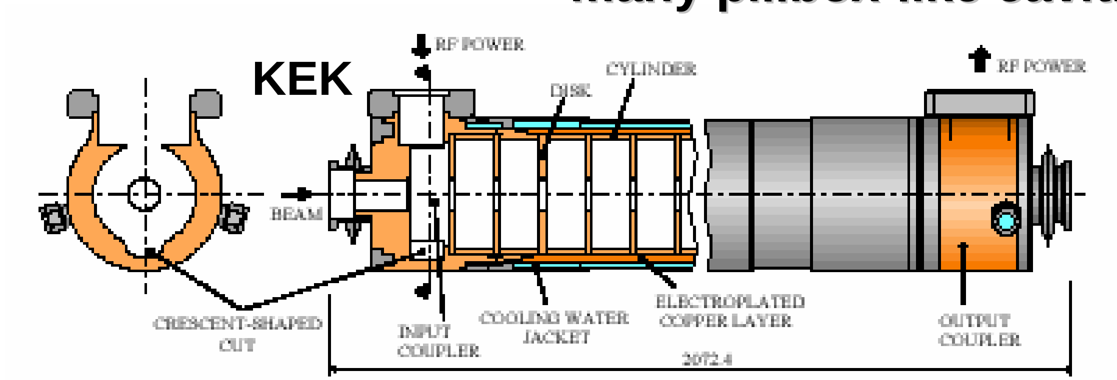


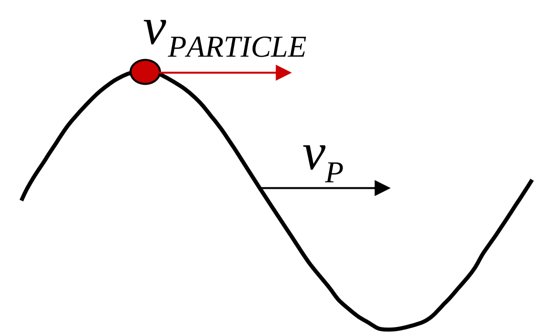
200 MHz RF source from radars

The Alvarez structures are still widely used as pre-accelerator for protons and ions. The particles at few hundred keV from a Cockcroft-Walton for example, are accelerated to few hundred MeV.

Accelerating Structure Evolution

Newer and more efficient RF structures were obtained by coupling together many pillbox-like cavities.





By "loading" the cylindrical structure by disks v_P can be reduced down to match the speed of the particle for an efficient acceleration

- Traveling wave, constant impedance: electric field decreases exponentially with length. The irises have constant radius.

- Traveling wave, constant gradient: the electric field is constant along the structure. The iris have decreasing radius.

- Standing wave: no wave propagation. The beam transit time in the cell must be much smaller than the wave period for efficient acceleration

- Normal conductive or Super conductive

L-band ~ 1.5 GHz, S-band ~ 3 GHz, X-band ~ 11GHz, ….

Energy Gain in RF Structures

$$E_{KIN} = |q| \sqrt{r_s L P_0} \sqrt{2\tau} \, \frac{1 - e^{-\tau}}{\tau} \cos\varphi_s$$

Traveling Wave
Constant Impedance

$$E_{KIN} = |q| \sqrt{r_s L P_0} \, \sqrt{1 - e^{-2\tau}} \, \cos\varphi_s$$

Traveling Wave
Constant Gradient

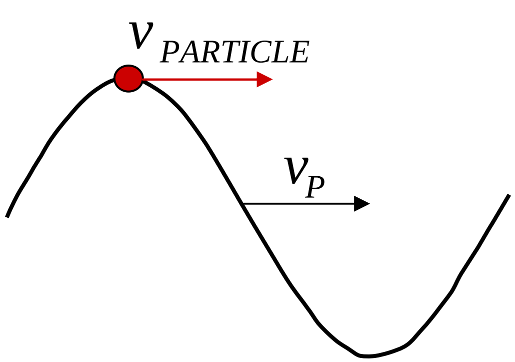
$$E_{KIN} = |q| \sqrt{R_s P_0} \, \frac{\sin\left(\dfrac{\omega \lambda_{RF}}{4 v_{particle}}\right)}{\dfrac{\omega \lambda_{RF}}{4 v_{particle}}} \quad \text{For a cavity with length } L = \frac{\lambda_{RF}}{2}$$

Standing
Wave

Transit
time factor

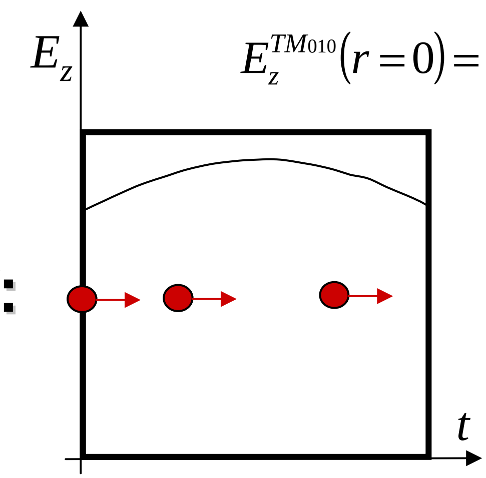
Synchronism:

TW:



SW:

$$E_z^{TM010}(r=0) = E_0 \cos(\omega_C t)$$



Scaling with Frequency

Choosing the RF frequency is a very critical step in designing an accelerator. Several parameters depend in opposite way on the frequency and the best match to the application will be a frequency that trading between contrasting requirements:

PARAMETER	Frequency Scaling	Frequency Preference
Shunt impedance per unit length (r_s)	$\omega^{1/2}$	High
Total RF peak power (P_0)	$\omega^{-1/2}$	High
Maximum possible electric field strength (empirical)	$\omega^{1/2}$	High
RF energy stored in the structure	ω^{-2}	High
Peak beam current at maximum conversion efficiency	$\omega^{-1/2}$	Low
Beam Loading ($-dV/di$)	$\omega^{1/2}$	Low
Maximum RF power available for single source	ω^{-2}	Low
Diameter of beam aperture	ω^{-1}	Low

$$E_{KIN} \propto \sqrt{r_s L P_0}$$

Higher frequencies pushes to high energy applications

Lower frequencies pushes to high current applications

The table is incomplete. For an exhaustive list see for example:
The Stanford Two-Mile Accelerator **(Chapter 6), R.B. Neal Editor (1968).**

Large and Small Linear Accelerators

• The combined development and optimization of the described RF structures jointly to the development of powerful and efficient RF sources (klystrons), permitted the ambitious design and construction of large and very high energy linear accelerators.

• Above all, we want to mention the 3-km linear accelerator that started operating in 1966 at the Stanford Linear Accelerator Center and that is capable of accelerating electron and positrons up to more than 50 GeV, with an average gradient in the RF structure of ~ 17 MeV/m.

•R&D on higher frequency RF structures is demonstrating gradients larger than 100 MeV/m.

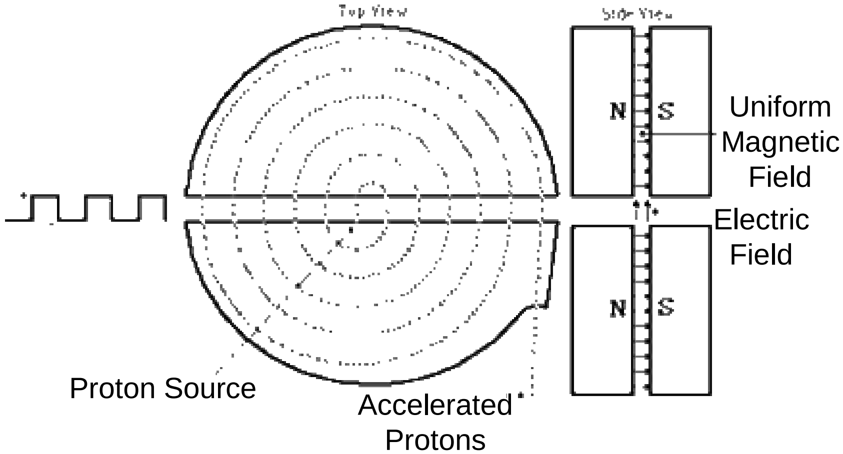
• In the International Linear Collider (ILC) project, electron and positron linacs longer than 30 Km and with energies over 500 GeV are under consideration.

• At the same time, much smaller linacs from few MeV to few hundred MeV are the "backbone" of the injector in most existing electron accelerators.

Linear Accelerators vs. Circular Accelerators

- RF cavity technology allowed the development of both linear and circular accelerators.
- The main advantage of circular accelerators is that a single cavity, where the beam passes many times guided by the confinement action of magnetic fields, is capable of very high energy acceleration. This is a very efficient scheme where only a relatively small amount of RF power is required.
- Unfortunately, for light particles the emission of synchrotron radiation can limit the maximum energy achievable (~ 100 GeV for electrons).
- In general, circular accelerators are more efficient with heavy particles and medium energy electrons, while linear accelerators are preferred with high energy electrons.
- Efficiency is not all. For example, circular machines usually show more stable beam characteristics while the beam emittance can be (maintained) smaller in linear accelerators. Different applications can find their best match in either one or the other schemes.

Cyclotron and Synchro-cyclotron




The first cyclotron
4.5” diameter (1929).


E. O. Lawrence
1939 Nobel Prize

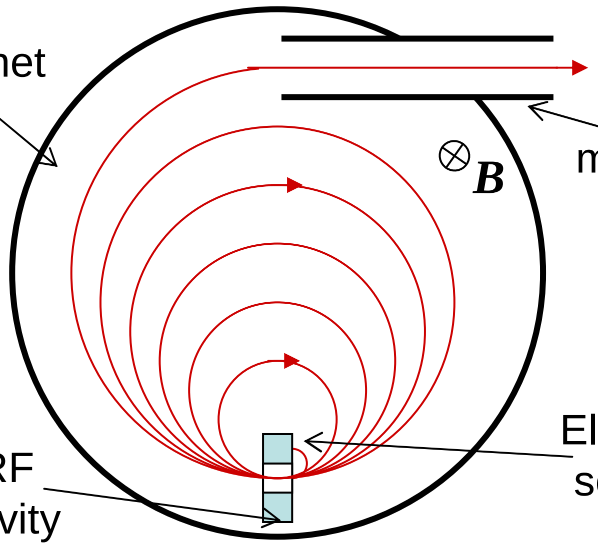
In an uniform magnetic field:

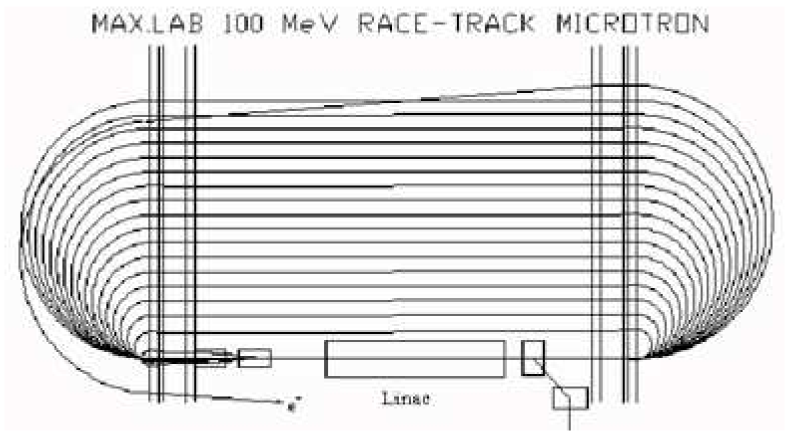
$$T_R = \frac{2\pi\, r}{v} = \frac{2\pi\, p}{veB} = \frac{2\pi\, mv}{veB} = \frac{2\pi\, m}{eB} \qquad for \quad v << c$$

For non-relativistic particles the revolution period does not depend on energy

- **If the RF frequency is equal to the particles revolution frequency synchronicity is obtained and acceleration is achieved.**
- **The synchro-cyclotron is a variation that allows acceleration also of relativistic particles. The RF frequency is dynamically changed to match the changing revolution frequency of the particle**
- **In 1946 Lawrence built in Berkeley the 184” synchro-cyclotron with an orbit radius of 2.337 m and capable of 350 MeV protons. The largest cyclotron still in operation is in Gatchina and accelerates protons to up 1 GeV for nuclear physics experiments.**

Microtrons





Moroz and Roberts 1958

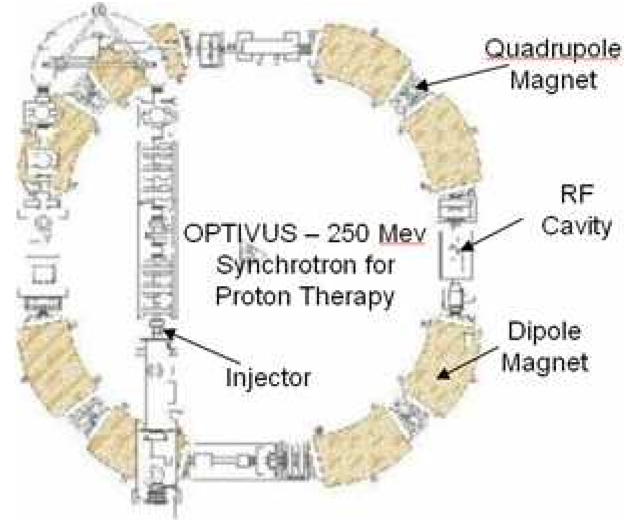
Synchronicity condition (energy gain per turn) → $\Delta\gamma = n \quad n \text{ is an integer}$

0.511 MeV for electrons
938 MeV for protons

Useful only for accelerating electrons.
The maximum energy is~ 30 MeV (limited by the magnet size)

Synchrotrons & Storage Rings

**•Achieving higher energies in cyclotrons requires very large magnets.
Above ~ 400 MeV the realization of cyclotrons becomes inconvenient and expensive**



- **In a synchrotron the radius is fixed and all the fields can be confined only around the orbit.**

$$\boxed{R = \frac{\gamma m_0 \beta c}{ZeB} = \text{constant} \;\Rightarrow B \text{ must scale} \propto \text{ to } \beta\gamma}$$

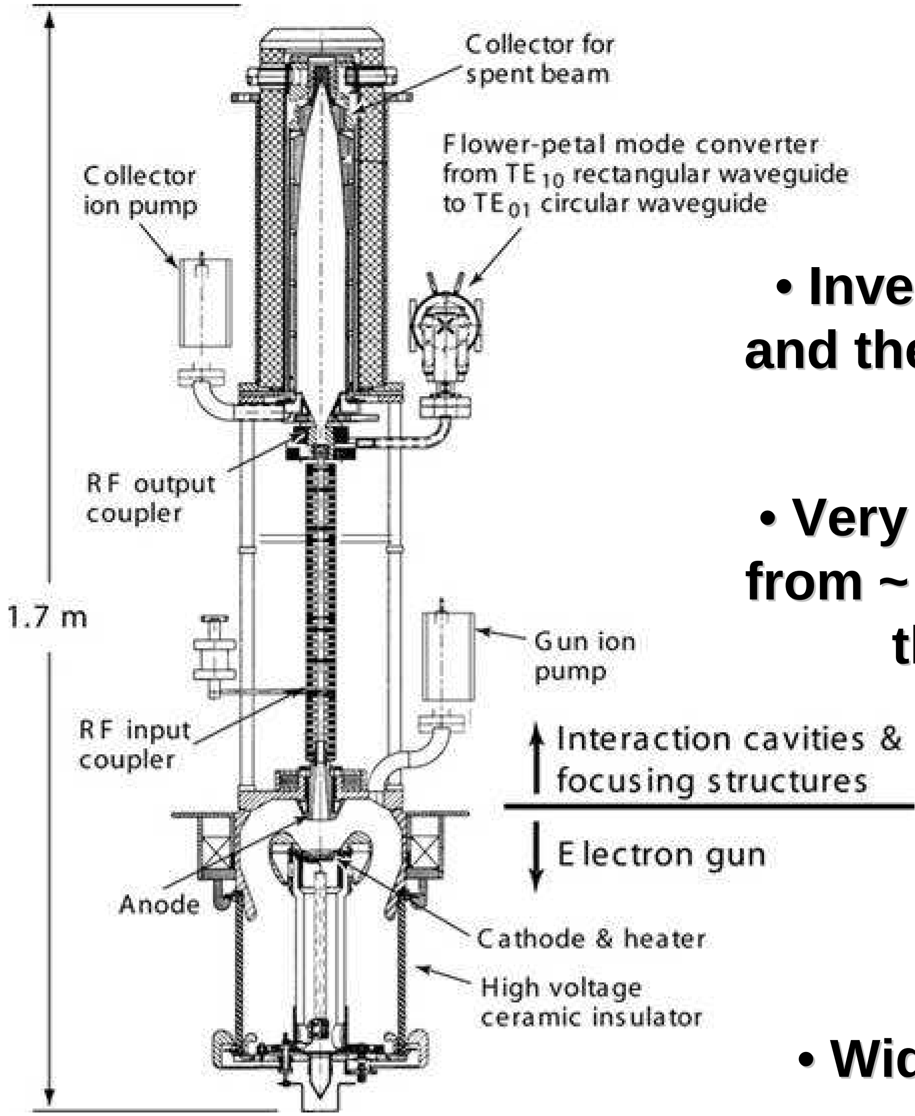
- **The synchronicity condition is given by:**

$$\boxed{f_{RF} = h\, f_0 = h\, \frac{Z\,e\,B}{2\pi\; m_0\; \gamma} \;\Rightarrow\; \begin{array}{c}\text{for non-relativistic beam } f_{RF} \\ \text{must change during acceleration}\end{array}}$$

- **Synchrotrons have achieved energy as high as 100 GeV for electrons and 1000 GeV for protons**

- **A storage ring is a synchrotron were the particles are not accelerated but just stored at a fixed energy for a relatively long time.
Colliders, synchrotron light sources, …**

RF Sources: The Klystron

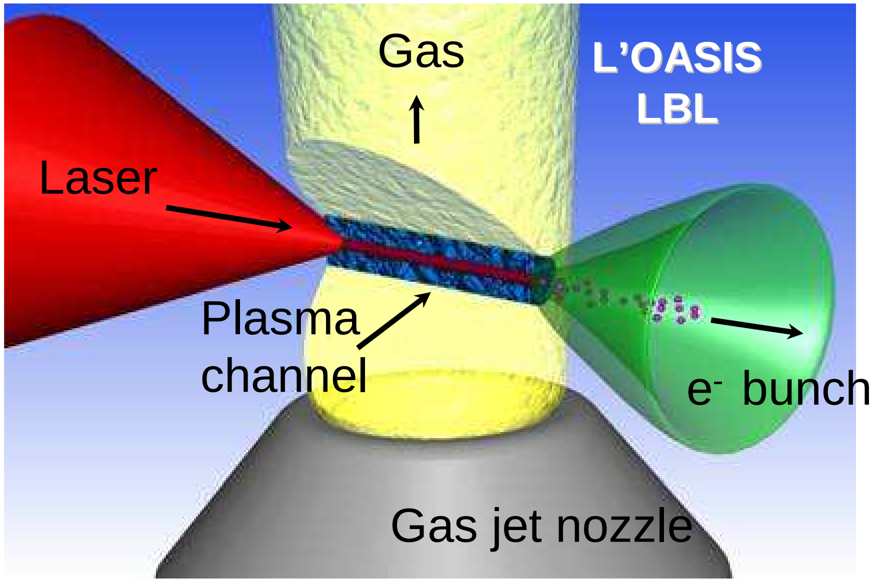


- **Invented by Hansen and the Varian brothers in 1937**
- **Very powerful source from ~ 100 MHz to more than 10 GHz**

- **Widely used in all kinds of accelerators**

R&D on Future Accelerator Schemes

• The R&D on new acceleration techniques is extremely active and addressed towards a very large variety of new accelerating techniques. Results are promising especially from the accelerating gradient point of view were extremely high values have been already obtained.



• Among the techniques under study, here we want to mention as an example, the one based on the so-called laser wakefield acceleration.

• A high intensity laser is focused on an atomic gas jet.

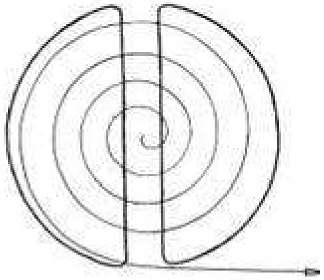
•The laser ionizes most of the atoms creating a plasma and also stimulating a resonant motion of the electrons in the plasma.

• This electron motion breaks the charge balance inside the very dense plasma inducing extremely high gradients in the plasma area surrounding the laser.

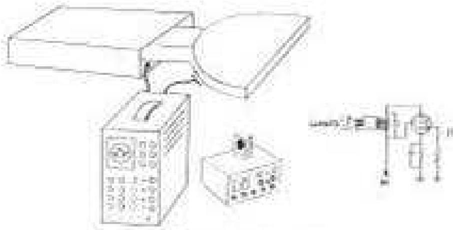
•Electrons in the plasma can find the right phase and can be accelerated to high energies. Gradients of many tens of GeV/m in few mm have been already demonstrated.

The Cyclotron: Different Points of View

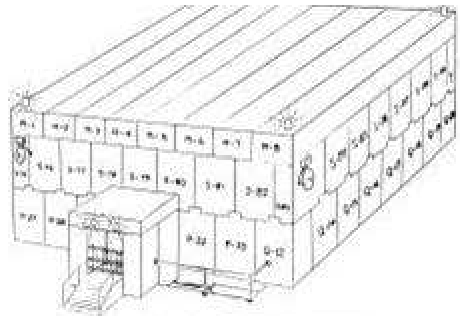
The Cyclotron, as seen by...



. . . the inventor

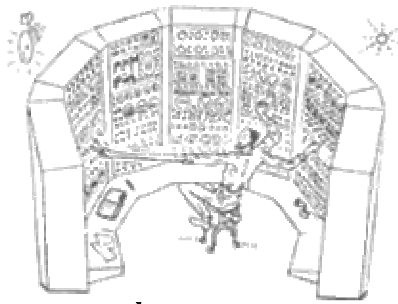


. . . the electrical engineer



. . . the health physicist

. . . the experimental physicist



…the operator

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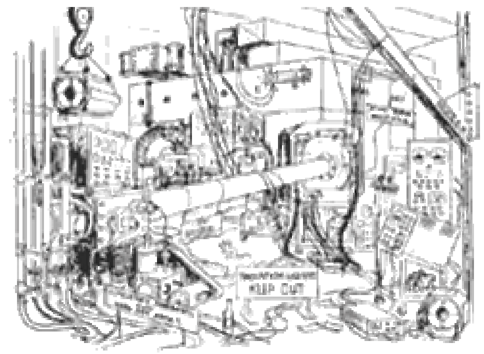
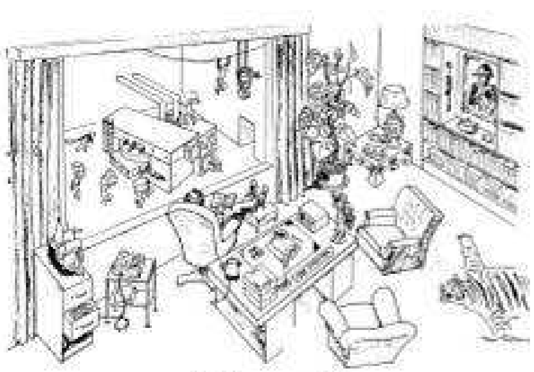
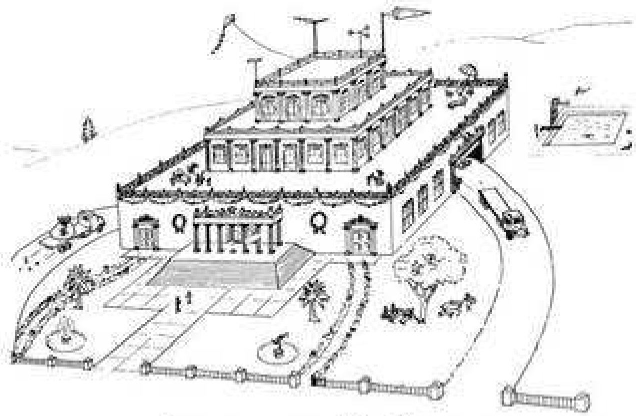
By Dave Judd and Ronn MacKenzie



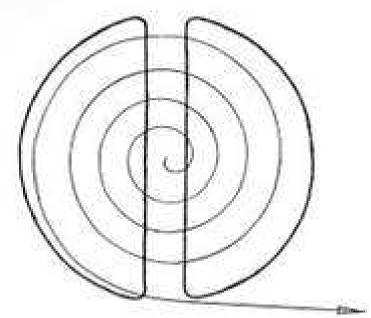
...the visitor



. . . the laboratory director



. . . the governmental funding agency



. . . the student

Possible Homework

- Prove that the magnetic field cannot do work
- Explain how the cascade generator in the Cockcroft-Walton schemes works
- Explain why the electric field inside the Van de Graaff sphere is zero
- Show that for free space electromagnetic waves E and B vectors are mutually orthogonal and have no component in the wave propagation direction. Show also the relation between their modules.
- Calculate the maximum energy gain in MeV vs. input power in MW for a 3.048 m long constant gradient accelerating structure with shunt impedance for unit length r_s= 53 MΩ/m and attenuation factor τ = 0.57. Calculate also the power required for accelerating relativistic electrons to 60 MeV
- Derive the expression for the transit time factor for a pillbox resonating in its TM_{010} mode
- Calculate the internal diameter of the external pipe of a 200 MHz Alvarez structure operating in the TM010 mode.

Vectorial Algebra

$$\nabla\cdot\left(\nabla\times\overline{F}\right)=0 \quad \forall\overline{F} \qquad\qquad \nabla\times\left(\nabla\times\overline{F}\right)=\nabla\left(\nabla\cdot\overline{F}\right)-\nabla^2\overline{F} \quad \forall\overline{F}$$

$$\nabla\times\left(\nabla u\right)=0 \qquad \forall u$$

or

$$\nabla\times\overline{F}=0 \quad \Leftrightarrow \quad \overline{F}=\nabla u$$

($\boldsymbol{F}$ is conservative if curl $\boldsymbol{F}$ is zero)

Volume Integral

$$\int_S \overline{F}\cdot\overline{n}\ dS=\int_V \nabla\cdot\overline{F}\ dV$$

Divergence Theorem

Surface Integral (Flux)

$$\oint_l \overline{F}\cdot d\overline{l}=\int_S\left(\nabla\times\overline{F}\right)\cdot\overline{n}\,dS$$

Curl Theorem (Stoke's Theorem)

Line Integral (Circuitation)

From Maxwell Equations to Wave Equation

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}$$

$\mathbf{E} = \mathbf{B} = 0$ is a solution, but there might be other solutions as well. Let us employ a useful identity from vector calculus.

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Where $\mathbf{A}$ can be any vector function. Taking the curl of the curl equations and applying the identity, we get the following.

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial^2 t}\mathbf{E}$$
$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial^2 t}\mathbf{B}$$